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Distributed Learning of Optimal Controls for Linear Systems

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Abstract— **While classic controller design methods rely on a model of the underlying dynamics, data-driven methods allow to compute controllers leveraging solely a set of previously recorded input-output trajectories, with relatively mild assumptions. Assuming knowledge of the dynamics is especially unrealistic in decentralized systems, since information is typically localized by design. In this paper we investigate a decentralized data-driven approach to learn quadratically-optimal controls for interconnected linear systems. Our main result is a distributed algorithm that computes a control input to reach a desired target configuration with provable, and tunable, suboptimality guarantees. Our distributed procedure converges after a finite number of iterations and the suboptimality gap can be characterized analytically in terms of the data properties. Our algorithm relies on a new set of closed-form data-driven expressions of quadratically-optimal controls, which complement the existing literature on data-driven linear-quadratic control. We complement and validate our theoretical analysis by means of numerical simulations with different interconnected systems.**

I. Introduction

With ever improving capabilities in sensor accuracy, storage efficiency, and processing power, data-driven tools to design controllers for systems with unknown and complex dynamics have seen a spike in popularity. These approaches depart from the classic system identification plus control design approach [1], as they allow to synthesize a controller directly from a collection of recorded controlled trajectories, bypassing the identification step altogether. Successes, and failures, of this data-enabled approach to control are numerous, e.g., [2]–[5], justifying the ongoing research interest.

Differently from most studies, which have focused on centralized data-driven problems where all data is processed at a central location, in this paper we study a data-driven control problem for an interconnected system, where local subsystems have access to only a subset of the available data. In this case, since both a model of the dynamics as well as the complete set of experimental data are not available at any particular location, subsystems must cooperate to collaboratively process the experimental data and obtain suitable control inputs. This distributed scenario is particularly appropriate for large network systems, where the dimensionality and distributed nature of the system challenge the collection of data from geographically sparse sensors, and whenever the behavior of certain parts of the system should remain private, thus preventing the use of global experimental data.

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Related work. A large body of research on data-driven control has focused on noiseless linear time-invariant dynamics and centralized settings. In particular, a number of works in this area have proposed data-based solutions to several control problems for linear systems, including open-loop optimal control [3], [6]–[8], closed-loop and robust control [4], [9]–[11], and predictive control [12], [13]. Recently, the more challenging nonlinear and noisy case has been also addressed in, e.g., [14]–[18]. However, to the best of our knowledge, the computation of data-driven controllers for decentralized systems is still a largely unexplored area, with a few notable exceptions [5], [19]. The work most similar to ours is [5], where the authors propose a distributed data-driven solution to the finite-horizon linear-quadratic problem based on a primal-dual distributed optimization algorithm. In our paper, we adopt a similar setting but use a different distributed approach to find a data-based solution to the problem. Differently from [5], our approach relies on a new set of closed-form expressions of optimal controls, and provably converges to a feasible solution in a finite number of steps, bounded by the diameter of the interconnection graph.

Paper contribution. The main contribution of this paper is a distributed algorithm for the design of optimal controls for an interconnected system with unknown dynamics. Our approach is data-driven, as it only relies on previously recorded input-output trajectories of the system. First, we derive novel closed-form data-driven expressions of quadratically-optimal controls to drive a linear system to a desired final state. Then, we present a distributed algorithm based on iterative projections for the interconnected subsystems to compute such controls, using only locally-collected experimental data and local interactions. A desirable property of our approach is that it converges in a finite number of iterations, and that its suboptimality gap can be characterized and made arbitrarily small as a function of the available data and parameters.

Paper organization. The paper is organized as follows. In Sec. II we describe the considered system and introduce the problem setup. In Sec. III we derive closed-form solutions to the problem of designing optimal data-driven controls, and present our distributed algorithm. We validate our results in Sec. IV, and leave all proofs in the Appendix.

Notation. Let $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the set of real, positive real, and non-negative real numbers, respectively. Given a matrix $A \in \mathbb{R}^{n\times m}$, $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$ denote the image and kernel of A. Given a subspace $\mathcal{V}$, we write $\mathrm{Basis}(\mathcal{V})$ to denote any full rank matrix whose columns span the subspace

$\mathcal{V}$. For any non-square matrix A, we denote its Moore-Penrose pseudoinverse as $A^\dagger$ and its transpose as $A^\top$. We let blk-diag$(A_1, \dots, A_n)$ be the block diagonal matrix with blocks $A_i \in \mathbb{R}^{k_i \times k_i}$. We let $A \succ 0$ ($A \succeq 0$) denote a positive definite (positive semidefinite) matrix, and $A^{1/2}$ the square root of the positive semidefinite matrix A. Finally, given vectors $x_i \in \mathbb{R}^{n_i}$, $i \in \{1, 2, \dots, \ell\}$, we let $\text{col}(x_1, x_2, \dots, x_\ell) = \begin{bmatrix} x_1^\top & x_2^\top & \cdots & x_\ell^\top \end{bmatrix}^\top \in \mathbb{R}^{n_1+n_2+\cdots+n_\ell}$.

II. Problem Setup and Preliminary Results

We study a data-driven control problem for an interconnected system comprising M subsystems with dynamics

$$x_i(t+1) = A_{ii}x_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij}x_j(t) + B_i u_i(t), \quad (1)$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state and input of subsystem i, respectively, and A_{ij}, B_i are constant matrices of appropriate dimension, with $i, j \in \{1, \dots, M\}$. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the (directed) interconnection graph among the subsystems, where $\mathcal{V} = \{1, \dots, M\}$ and $\mathcal{E} = \{(i,j) : A_{ij} \neq 0\}$, and let $\mathcal{N}_i$ be the neighboring set of node i, i.e., $j \in \mathcal{N}_i \iff A_{ij} \neq 0$. Notice that the interconnection matrices A_{ij}, as well as the local input matrices B_i, are not required to be nonzero. See Fig. 1 for an illustrative example.

Let $\sum_{i=1}^M n_i = n$ and $\sum_{i=1}^M m_i = m$, and notice that

$$\underbrace{\begin{bmatrix} x_1(t+1) \\ \vdots \\ x_M(t+1) \end{bmatrix}}_{x(t+1)} = \underbrace{\begin{bmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & \ddots & \vdots \\ A_{M1} & \cdots & A_{MM} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ \vdots \\ x_M(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} B_1 & 0 & 0 \\ & \ddots & \\ 0 & 0 & B_M \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1(t) \\ \vdots \\ u_M(t) \end{bmatrix}}_{u(t)} \quad (2)$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the dynamics and input matrices of the interconnected system. We assume that (A, B) is controllable in T-steps [20] and that $x(0) = 0$.

We study the following finite-time optimal control problem with quadratic cost function and terminal constraint:

$$\min_{u_1,\dots,u_M} \quad \sum_{t=0}^{T-1}\sum_{i=0}^{M} x_i^\top(t)Q_i^\top Q_i x_i(t) + u_i^\top(t)R_i^\top R_i u_i(t) \quad (3)$$

$$\text{subject to} \quad x_i(t+1) = A_{ii}x_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij}x_j(t) + B_i u_i(t)$$

$$x_i(T) = x_i^{\text{f}}, \text{ with } i, j \in \{1, \dots, M\},$$

where Q_i and R_i are arbitrary matrices of appropriate dimension with $R_i^\top R_i \succ 0$, for all $i \in \{1, \dots, M\}$.[1] If the system matrices are known, the solution to problem (3) can be readily derived.[2] To see this, let

$$\underbrace{\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(T) \end{bmatrix}}_{X_T} = \underbrace{\begin{bmatrix} B & \cdots & 0 & 0 \\ AB & \cdots & 0 & 0 \\ & \ddots & & \\ A^{T-1}B & \cdots & AB & B \end{bmatrix}}_{F_T} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(T-1) \end{bmatrix}}_{U_T}, \quad (4)$$

and let $x(T) = x^{\text{f}} = \text{col}(x_1^{\text{f}}, \dots, x_M^{\text{f}})$ and

$$C_T = \begin{bmatrix} A^{T-1}B & \cdots & AB & B \end{bmatrix}. \quad (5)$$

Using (4)-(5) and substituting the constraints into the cost function, the minimization problem (3) can be rewritten as

$$\min_{w} \quad \left\| P^{1/2}(C_T^\dagger x^{\text{f}} + K_C w) \right\|_2^2, \quad (6)$$

where $K_C = \text{Basis}(\text{Ker}(C_T))$, $P = F_T^\top Q^\top Q F_T + R^\top R$,

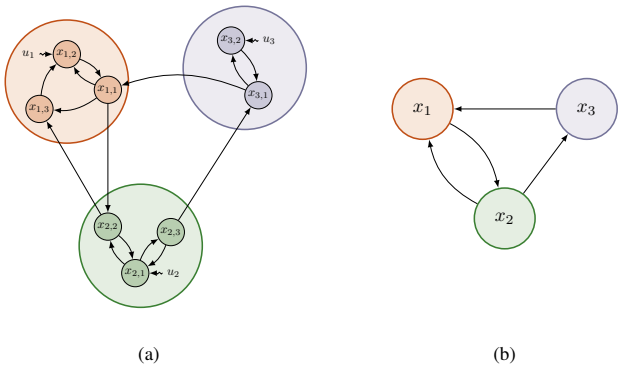
$$R = \text{blk-diag}(\underbrace{R_1, \dots, R_M}_{\text{time } 0}, \dots, \underbrace{R_1, \dots, R_M}_{\text{time } T-1}), \text{ and}$$
$$Q = \text{blk-diag}(\underbrace{Q_1, \dots, Q_M}_{\text{time } 0}, \dots, \underbrace{Q_1, \dots, Q_M}_{\text{time } T-1}). \quad (7)$$

Finally, computing the minimizer of (6) yields

$$U_T^* = (I - K_C(P^{1/2}K_C)^\dagger P^{1/2})C_T^\dagger x^{\text{f}}, \quad (8)$$

which equals the unique minimizer of (3).

In this paper we seek a solution to the minimization problem (3) when the system matrices are not known and, instead, a set of local control experiments is available to the subsystems. In particular, we assume that the subsystems have completed a set of N experiments where, starting from the origin, each subsystem injects (non-optimal and possibly



(a) (b)

Fig. 1. This figure shows an interconnected system with three subsystems, as described in (2). Subsystems 1 and 2 consist of $n_i = 3$ states each, while subsystem 3 has $n_i = 2$ states. All subsystems have one input, i.e., $u_i = 1$. Panel (a) describes the interconnected system. Panel (b) describes the interconnection graph $\mathcal{G}$ among subsystems.

[1] We assume the matrices Q_i and R_i to be constant, although our analysis applies directly also to the case where these matrices are time varying.

[2] Problem 3 is convex and feasible, because (A, B) is controllable. Thus, a unique solution always exists.

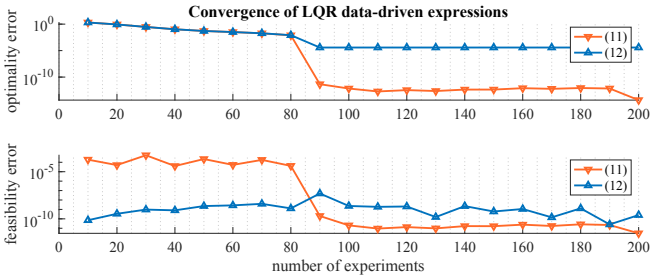


Fig. 2. This figure shows the convergence of data-driven expressions (11) and (12) to the model based solution (8). The system interconnections are described in Fig. 1 (matrices A_{ij} are chosen from a random distribution). In this example, Assumption 2.1 is satisfied from $T = 90$, after which the optimal solution is reached. The optimality error is the two-norm difference of the cost function $J(x, u)$ in (8) and (11), (resp. (8) and (12)), and the feasibility error is the two-norm of distance from the desired final state.

random) control sequences of length T and records its state trajectory. The data recorded by the i-th subsystem is

$$\begin{aligned} X_i &= \begin{bmatrix} X_i^1 & \cdots & X_i^N \end{bmatrix}, U_i = \begin{bmatrix} U_i^1 & \cdots & U_i^N \end{bmatrix}, \text{ and} \\ Y_i &= \begin{bmatrix} Y_i^1 & \cdots & Y_i^N \end{bmatrix}, \end{aligned} \tag{9}$$

where X_i contains the state trajectories, U_i the control sequences, and Y_i the final states of subsystem i during the N experiments. Specifically, $\text{col}(x_i(1), \dots, x_i(T)) = X_i^j$, with $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$, is the state trajectory of subsystem i when $x_k(0) = 0$ and $\text{col}(u_k(0), \dots, u_k(T-1)) = U_k^j$, for all $k \in \{1, \dots, M\}$. Further, $Y_i^j = x_i(T)$ after the j-th experiment. Finally, we indicate with

$$\begin{aligned} X &= \begin{bmatrix} X_1^\top & \cdots & X_M^\top \end{bmatrix}^\top, U = \begin{bmatrix} U_1^\top & \cdots & U_M^\top \end{bmatrix}^\top, \text{ and} \\ Y &= \begin{bmatrix} Y_1^\top & \cdots & Y_M^\top \end{bmatrix}^\top. \end{aligned} \tag{10}$$

the matrices comprising data recorded from all subsystems. Throughout the paper we make the following assumption.

Assumption 2.1: The input matrix U is of full row rank.

Assumption 2.1 ensures that any desirable control belongs to the image of the recorded data, i.e., for any T-steps input sequence $u \in \mathbb{R}^{mT}$, there exists $\alpha \in \mathbb{R}^N$ such that $u = U\alpha$.

We remark that each subsystem knows only its recorded data, and is unaware of the system matrices as well as the experimental data recorded by the other subsystems. In the next sections we characterize a distributed algorithm for the subsystems to cooperatively compute the optimal solution to (3) using only the local data (9).

III. Distributed Learning of Optimal Controls

When the system matrices are not known, a solution to the minimization problem (3) can be computed using the experimental data collected by all subsystems.

Theorem 3.1: ***(Data-driven solution to optimal control problem*** (3)*)* Let $P = (XU^\dagger)^\top Q^\top Q(XU^\dagger) + R^\top R$, where X and U are as in (10), and Q and R as in (7). Then,

$$U_T^* = P^{-\frac{1}{2}}(YU^\dagger P^{-\frac{1}{2}})^\dagger x^{\text{f}} \tag{11}$$

equals the solution of the minimization problem (3). □

Theorem 3.1 provides a data-driven expression of the optimal control input for the minimization problem (3.1), and it complements the explicit formulas presented in [6] for the minimum-energy case (that is, when $Q_i = 0$). The expression (11) can equivalently be written as (see also [3])

$$U_T^* = U\left(I - K_Y\left(\begin{bmatrix} RU \\ QX \end{bmatrix} K_Y\right)^\dagger \begin{bmatrix} RU \\ QX \end{bmatrix}\right) Y^\dagger x^{\text{f}}, \tag{12}$$

with $K_Y = \text{Basis}(\text{Ker}(Y))$, which further highlights the dependency of the optimal controls on the data (see the Appendix for a proof of these results), and is the data-driven formula analogous to (8), as highlighted in Fig. 2. Yet, both expressions (11) and (12) require all the experimental data, as well as knowledge of all local cost functions, which is undesirable for interconnected and possibly large systems. While a distributed data-driven solution to the minimization problem (3) could be obtained using standard techniques for distributed optimization, e.g., see [5], we follow a different approach that will lead to an algorithm with desirable convergence properties.

Following [6] we define the following auxiliary problems:

$$\begin{aligned} \arg\min_{\alpha} \quad & \left\| \begin{bmatrix} RU \\ QX \end{bmatrix} \alpha \right\|_2^2 \\ \text{s.t.} \quad & Y\alpha = x^{\text{f}} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \arg\min_{\alpha} \quad & \left\| \begin{bmatrix} \alpha \\ v \\ w \end{bmatrix} \right\|_2^2 \\ \text{s.t.} \quad & \begin{bmatrix} Y & 0 & 0 \\ RU & \varepsilon I^u & 0 \\ QX & 0 & \varepsilon I^x \end{bmatrix} \begin{bmatrix} \alpha \\ v \\ w \end{bmatrix} = \begin{bmatrix} x^{\text{f}} \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{14}$$

where $\varepsilon \in \mathbb{R}_{>0}$ is a tunable parameter and I^u, I^x are identity matrices of compatible dimensions.

We now characterize the feasibility and optimality properties of the auxiliary problems (13) and (14).

Lemma 3.2: ***(Properties of the auxiliary problem*** (13)*)*
If α^* is a minimizer of Problem (13), then $U_T^* = U\alpha^*$ is the minimizer of Problem (3). □

Lemma 3.3: ***(Properties of the auxiliary problem*** (14)*)*
The minimization problem (14) is feasible, and admits a unique solution, when $\varepsilon \neq 0$. Furthermore, if $\alpha^*(\varepsilon)$, $v^*(\varepsilon)$ and $w^*(\varepsilon)$ are the minimizers of Problem (14) for $\varepsilon > 0$, then

$$\lim_{\varepsilon \to 0^+} U\alpha^*(\varepsilon) = U_T^*, \tag{15}$$

where U_T^* is the minimizer of Problem (3). □

Lemma 3.4: ***(Convergence of*** (15)*)*
Let $\alpha^*(\varepsilon)$ be the minimizer to Problem (14). It holds

$$\delta(\varepsilon) = \|U_T^* - U\alpha^*(\varepsilon)\| = \left\|UK_Y Z(\varepsilon) K_Y^\top H^\top H Y^\dagger x^{\text{f}}\right\|, \tag{16}$$

where $Z(\varepsilon) = (K_Y^\top(\varepsilon^2 I + H^\top H)K_Y)^\dagger - (K_Y^\top H^\top H K_Y)^\dagger$, and $H = \begin{bmatrix} (RU)^\top & (QX)^\top \end{bmatrix}^\top$. □

Lemma 3.2 and 3.3 provide alternative data-driven ways of computing the solution to the optimal control problem (3), which constitute the basis of our distributed procedure.

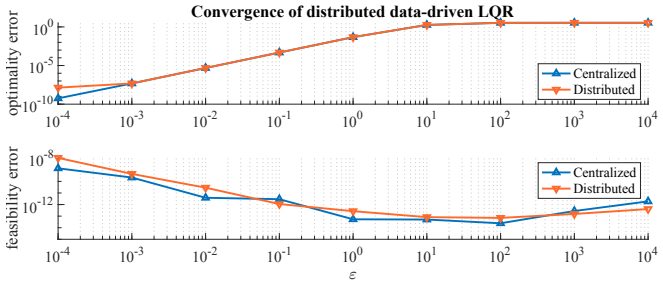


Fig. 3. This figure shows the convergence of the data-driven solutions to (14) to the model based expression (8). Upward triangles show the convergence of the centralized approach (15), while downward triangles show the convergence of the distributed Algorithm 1. The system interconnections are described in Fig. 1 (matrices A_{ij} are chosen from a random distribution), and the optimality and feasibility errors are described in Fig. 2.

Algorithm 1: Distributed data-driven optimal control

Input: x_i^{f}, W_i, d
$\hat{\gamma}_i = W_i^{\dagger}\,\mathrm{col}(x_i^{\mathrm{f}}, 0_i^v, 0_i^w)$
$K_i = \mathrm{Basis}(\mathrm{Ker}(W_i))$
for d *times* **do**
  for $j \in \mathcal{N}_i$ **do**
    Receive: $\hat{\gamma}_j$, K_j
    $\hat{\gamma}_i^+ = \hat{\gamma}_i + \begin{bmatrix} K_i & 0 \end{bmatrix} \begin{bmatrix} -K_i & K_j \end{bmatrix}^{\dagger} (\hat{\gamma}_i - \hat{\gamma}_j)$
    $K_i = \mathrm{Basis}(\mathrm{Im}(K_i) \cap \mathrm{Im}(K_j))$
    $\hat{\gamma}_i = \hat{\gamma}_i^+$
  end
  Transmit: $\hat{\gamma}_i$, K_i
end
Return: $\hat{\alpha}_i = \hat{\gamma}_i[1:N]$

Informally, our distributed data-driven procedure to compute the optimal controls is as follows:

(S1) At the algorithm initialization each subsystem i computes $\hat{\gamma}_i = \mathrm{col}(\hat{\alpha}_i, \hat{v}_i, \hat{w}_i)$ as the minimum norm solution to $W_i\hat{\gamma}_i = \mathrm{col}(x_i^{\mathrm{f}}, 0_i^v, 0_i^w)$, that is $\hat{\gamma}_i = W_i^{\dagger}\,\mathrm{col}(x_i^{\mathrm{f}}, 0_i^v, 0_i^w)$, where $0_i^v \in \mathbb{R}^{Tm_i}$ and $0_i^w \in \mathbb{R}^{Tn_i}$ are vectors of all zeros. Let

$$W_i = \begin{bmatrix} Y_i & 0 & 0 \\ R_iU_i & \varepsilon I_i^u & 0 \\ Q_iX_i & 0 & \varepsilon I_i^x \end{bmatrix}, \tag{17}$$

and $K_i = \mathrm{Basis}(\mathrm{Ker}(W_i))$, whose columns span the subspace orthogonal to $\hat{\gamma}_i$. Here, I_i^u (resp. I_i^x) is a matrix whose rows are the rows of I^u (resp. I^x) corresponding to the indexes that extract R_iU_i (resp. Q_iX_i) from RU (resp. QX) in (14), i.e., $R_iU_i = I_i^u RU$ (resp. $Q_iX_i = I_i^x QX$).

(S2) Each subsystem i transmits $\hat{\gamma}_i$ and K_i to its neighboring subsystem j, and receives its neighbors' $\hat{\gamma}_j$ and K_j.

(S3) Every subsystem i updates its current estimate $\hat{\gamma}_i$ as

$$\hat{\gamma}_i^+ = \hat{\gamma}_i + \begin{bmatrix} K_i & 0 \end{bmatrix} \begin{bmatrix} -K_i & K_j \end{bmatrix}^{\dagger} (\hat{\gamma}_i - \hat{\gamma}_j), \tag{18}$$

and updates the space orthogonal to the newly found $\hat{\gamma}_i$ as $K_i = \mathrm{Basis}(\mathrm{Im}(K_i) \cap \mathrm{Im}(K_j))$. Finally, the first N elements of $\hat{\gamma}_i$ are returned, i.e., the elements corresponding to $\hat{\alpha}_i$.

At the initialization step (S1) the solution $\hat{\gamma}_i$ is feasible for each subsystem, i.e., $Y_i\hat{\alpha}_i = x_i^{\mathrm{f}}$, $i \in \{1, \dots, M\}$. This follows from the fact that all of the matrices Y_i are of full row rank because of Assumption 2.1 and the controllability of (A, B). However, at this stage, $\hat{\gamma}_i$, and hence $\hat{\alpha}_i$, is not optimal, as no information has yet been shared among the subnetworks. During each update in (S3) $\hat{\gamma}_i$ remains feasible and $\hat{\alpha}_i$ monotonically approaches the minimizer $\alpha^*(\varepsilon)$ to (14). If the interconnection graph $\mathcal{G}$ is connected, after at most d iterations, with d being the diameter of $\mathcal{G}$, the estimate $\hat{\alpha}_i$ of each subsystem equals the minimizer $\alpha^*(\varepsilon)$ to (14).

This procedure is described formally in Algorithm 1, and its convergence to the solution to (14) is proved next.

Theorem 3.5: ***(Distributed learning of data-driven optimal controls)*** Let $\mathcal{G}$ be a connected graph with diameter d, and let $\hat{\alpha}_i$ be the value returned by Algorithm 1. Then, for all $i \in \{1, \dots, M\}$, $\hat{\alpha}_i$ is the solution to (14). □

In Fig. 3 we show the convergence, as ε approaches zero, of $U\alpha^*(\varepsilon)$ to the optimal input U_T^*, for $\alpha^*(\varepsilon)$ computed as the centralized solution to (14) and for $\alpha^*(\varepsilon)$ computed via the distributed procedure in Algorithm 1. The solution is feasible for every choice of ε, as the feasibility error is always small. The error on the computed controls decreases with ε, following the error characterization of Lemma 3.4.

A key strength of this approach is given by its convergence properties. In fact, once a desired accuracy $\bar{\delta} > 0$ on the optimal solution is chosen, one can always find ε such that $\delta(\varepsilon) < \bar{\delta}$, guaranteeing the desired performance, through (16). We remark that the optimality gap (16) is a function of the available data and the parameters Q and R. Finally, Algorithm 1 reaches a feasible solution in a finite number of steps, which is bounded by the network's diameter. These are desirable properties of this approach, as it guarantees feasibility in a finite number of steps, and a bounded error on the optimality of the solution.

IV. Numerical results

We complement the theoretical results of Section III through numerical simulations. To highlight the convergence properties of Algorithm 1, we consider (see Fig. 4) a chain network and a ring network. Both networks have the same number of subnetworks, but differ in diameter. The chain (or line) network is intended as a *worst-case* scenario for this approach, as it maximizes the diameter of the network for a given number of subsystems. As is shown in Fig. 4, after a number of iterations equal to the diameter of the network we reach the optimal, feasible, solution. For the k-th iteration of Algorithm 1, subnetwork i computes its own control inputs as $\hat{u}_i[k] = U_i\hat{\alpha}_i[k]$, where $\hat{\alpha}_i[k]$ is the interim value of $\hat{\alpha}_i$ at iteration k. The dynamics of (2) when $\hat{u}_i[k]$ is fed to all $i \in \{1, \dots, M\}$ is compared to the one induced by the model-based optimal control (3). For each iteration we highlight the distance of the solution from the feasibility constraint and from the optimal cost. Finally, we perform the same analysis on a ring network which has the

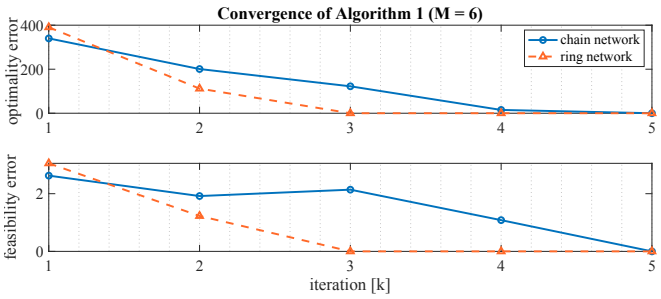


Fig. 4. This figure shows the convergence of Algorithm 1 ($\varepsilon = 0.01$) for an undirected chain network (solid line) and a ring network (dashed line). Both networks have $M = 6$ subsystems; the chain network has diameter $d = 5$, and the ring network has diameter $d = 3$. For both networks the total number of states and inputs, respectively, is $n = 24$ and $m = 14$.

same number of nodes as in the previous example, but has a smaller diameter. As expected, in this scenario Algorithm 1 converges to the solution in a smaller number of iterations, thanks to the network's smaller diameter.

V. Conclusions

In this paper we present a distributed algorithm to compute quadratically-optimal controls using only local experimental data, and without knowing a model of the underlying dynamics. Our algorithm relies on a new set of closed-form expressions of the control inputs that solve an optimal (open-loop and finite-time) control problem with quadratic cost function and terminal constraint, converges in finite-time, and provides guarantees on the optimality properties of the solution. Issues of robustness to noise and adversarial manipulation of the data and cooperation protocol, as well as the application to systems with nonlinear and time-varying dynamics, remain promising avenues for future investigation.

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Appendix

A. Proof of Theorem 3.1

Using the fact that $X_T = F_T U_T$ (cf. Eq. (4)), Problem (3) can be written in centralized form as

$$\begin{aligned} \min_{U_T} \quad & \left\| P^{1/2} U_T \right\|_2^2 \\ \text{s.t.} \quad & C_T U_T = x^{\mathrm{f}} \end{aligned} \tag{19}$$

where $P = F_T^\top Q^\top Q F_T + R^\top R \succ 0$. Letting $V_T = P^{1/2} U_T$, Problem (19) becomes

$$\begin{aligned} \min_{U_T} \quad & \|V_T\|_2^2 \\ \text{s.t.} \quad & C_T P^{-1/2} V_T = x^{\mathrm{f}} \end{aligned} \tag{20}$$

whose minimizer is $V_T^* = (C_T P^{-1/2})^\dagger x^{\mathrm{f}}$. Thus, the minimizer of (19) can be written as

$$U_T^* = P^{-1/2} V_T^* = P^{-1/2} (C_T P^{-1/2})^\dagger x^{\mathrm{f}}. \tag{21}$$

Next, note that matrices C_T and P can be written in terms of the data matrices U, X, and Y as

$$C_T = Y U^\dagger, \tag{22}$$

$$P = (X U^\dagger)^\top Q^\top Q (X U^\dagger) + R^\top R. \tag{23}$$

Substituting (22) and (23) in (21) concludes the proof. □

B. Proof of Lemma 3.2

By linearity of the system, the state trajectory and final state generated by the input vector $U\alpha$, $\alpha \in \mathbb{R}^n$, are $X\alpha$ and $Y\alpha$, respectively. Thus any minimizer α^* to Problem (13) yields the (unique) optimal control sequence via $U_T^* = U\alpha^*$. We can derive an explicit form of the optimal vectors α^*,

by substituting the constraint of Problem (13) into the cost function. This yields the problem

$$\min_{w} \quad \left\| \begin{bmatrix} RU \\ QX \end{bmatrix} (Y^\dagger x^{\mathrm{f}} + K_Y w) \right\|_2^2 \tag{24}$$

where $K_Y = \mathrm{Ker}(Y)$. The minimizers w^* to (24) define the set of optimal vectors α^* via

$$\begin{aligned} \alpha^* &= Y^\dagger x^{\mathrm{f}} + K_Y w^* \\ &= \left(I - K_Y \left(\begin{bmatrix} RU \\ QX \end{bmatrix} K_Y \right)^\dagger \begin{bmatrix} RU \\ QX \end{bmatrix} \right) Y^\dagger x^{\mathrm{f}} + r, \end{aligned} \tag{25}$$

where $r \in \mathrm{Ker} \begin{bmatrix} (RU)^\top & (QX)^\top \end{bmatrix}^\top \subseteq \mathrm{Ker}\, U$. The resulting (unique) optimal control $U_T^* = U\alpha^*$ is given in (12). □

C. Proof of Lemma 3.3

From the constraints in (14) it holds, for all $\varepsilon > 0$,

$$v = -\frac{1}{\varepsilon} RU\alpha, \quad w = -\frac{1}{\varepsilon} QX\alpha.$$

Substituting these equations in the cost function, Problem (14) can be rewritten as

$$\begin{aligned} \alpha^*(\varepsilon) = \arg\min_{\alpha} \quad & \varepsilon^2 \left\|\alpha\right\|_2^2 + \left\| \begin{bmatrix} RU \\ QX \end{bmatrix} \alpha \right\|_2^2 \\ \text{s.t.} \quad & Y\alpha = x^{\mathrm{f}}. \end{aligned} \tag{26}$$

If $\alpha^*(\varepsilon)$ is bounded as $\varepsilon \to 0^+$, then $\alpha^*(\varepsilon)$ converges to the (minimum norm) solution to Problem (13) as $\varepsilon \to 0^+$, and (15) holds. Thus, it remains to prove the boundedness of $\alpha^*(\varepsilon)$ as $\varepsilon \to 0^+$. To this end, we note that by writing $\alpha^*(\varepsilon) = \alpha_1(\varepsilon) + \alpha_2(\varepsilon)$ with

$$\alpha_1(\varepsilon) \perp \mathrm{Ker} \begin{bmatrix} RU \\ QX \end{bmatrix} \text{ and } \alpha_2(\varepsilon) \in \mathrm{Ker} \begin{bmatrix} RU \\ QX \end{bmatrix},$$

the cost of (26) evaluated at $\alpha^*(\varepsilon)$ reads as

$$C(\alpha^*(\varepsilon)) = \varepsilon^2 \left\|\alpha_1(\varepsilon)\right\|_2^2 + \varepsilon^2 \left\|\alpha_2(\varepsilon)\right\|_2^2 + \left\| \begin{bmatrix} RU \\ QX \end{bmatrix} \alpha_1(\varepsilon) \right\|_2^2. \tag{27}$$

The vector $\alpha_1(\varepsilon)$ satisfies $Y\alpha_1(\varepsilon) = Y\alpha^*(\varepsilon) = x^{\mathrm{f}}$ because $\mathrm{Ker} \begin{bmatrix} (RU)^\top & (QX)^\top \end{bmatrix}^\top \subseteq \mathrm{Ker}\, U \subseteq \mathrm{Ker}\, Y$. From the latter fact and (27), it follows that it must be $\alpha_2(\varepsilon) = 0, \forall \varepsilon > 0$, for $\alpha^*(\varepsilon)$ to be optimal. Further, from (27) and the fact that there always exists a α (e.g., $\alpha = Y^\dagger x^{\mathrm{f}}$) which is independent of ε and satisfies the constraint in (26) (thus yielding a cost which is bounded $\forall \varepsilon > 0$), it follows that $\alpha_1(\varepsilon) = \alpha^*(\varepsilon)$ must be bounded as $\varepsilon \to 0^+$. This concludes the proof. □

D. Proof of Lemma 3.4

Letting $H = \begin{bmatrix} U^\top R^\top & X^\top Q^\top \end{bmatrix}^\top$, Problem (14) can be rewritten as (cf. (26)),

$$\begin{aligned} \alpha^*(\varepsilon) = \arg\min_{\alpha} \quad & \left\| \begin{bmatrix} \varepsilon I \\ H \end{bmatrix} \alpha \right\|_2^2 \\ \text{s.t.} \quad & Y\alpha = x^{\mathrm{f}}. \end{aligned} \tag{28}$$

The minimizer of (28) is $\alpha^*(\varepsilon) = Y^\dagger x^{\mathrm{f}} - K_Y w^*(\varepsilon)$ with

$$\begin{aligned} w^*(\varepsilon) &= \arg\min_{w} \quad \left\| \begin{bmatrix} \varepsilon I \\ H \end{bmatrix} (Y^\dagger x^{\mathrm{f}} + K_Y w) \right\|_2^2 \\ &= \left(\begin{bmatrix} \varepsilon I \\ H \end{bmatrix} K_Y \right)^\dagger \begin{bmatrix} \varepsilon I \\ H \end{bmatrix} Y^\dagger x^{\mathrm{f}} \\ &= \left(K_Y^\top (\varepsilon^2 I + H^\top H) K_Y \right)^\dagger K_Y^\top (\varepsilon^2 I + H^\top H) Y^\dagger x^{\mathrm{f}} \\ &= \left(K_Y^\top (\varepsilon^2 I + H^\top H) K_Y \right)^\dagger K_Y^\top H^\top H Y^\dagger x^{\mathrm{f}}, \end{aligned} \tag{29}$$

where in the last-but-one step we used that $A^\dagger = (A^\top A)^\dagger A^\top$ for any matrix A, and in the last step $K_Y^\top Y^\dagger = 0$. Similarly, we can rewrite any minimizer to Problem (13) as (cf. (25))

$$\begin{aligned} \alpha^* &= \left(I - K_Y \left(H K_Y \right)^\dagger H \right) Y^\dagger x^{\mathrm{f}} + r \\ &= \left(I - K_Y \left(K_Y^\top (H^\top H) K_Y \right)^\dagger K_Y^\top H^\top H \right) Y^\dagger x^{\mathrm{f}} + r, \end{aligned} \tag{30}$$

where $r \in \mathrm{Ker}(H)$. From (28) and (30), it follows that $\|U(\alpha^* - \alpha^*(\varepsilon))\| = \|U_T^* - U\alpha^*(\varepsilon)\|$ equals (16).

E. Proof of Theorem 3.5

The proof follows an argument similar to the one of [21, Theorem 3.3]. Let $\hat{\gamma}_i$ be the estimate of subsystem i, W_i be defined as in (17), and $K_i = \mathrm{Basis}(\mathrm{Ker}(W_i))$. Observe that $\hat{\gamma} = W_i^\dagger \mathrm{col}(x_i^{\mathrm{f}}, 0_i^v, 0_i^w) \perp \mathrm{Ker}(W_i)$. Let i and j be two neighboring subsystems, i.e, $(i,j) \in \mathcal{E}$, then there exist two vectors v_i and v_j such that $\hat{\gamma}_i + K_i\kappa_i = \hat{\gamma}_j + K_j\kappa_j$. In particular, such vectors can be chosen as

$$\begin{bmatrix} \kappa_i \\ \kappa_j \end{bmatrix} = \begin{bmatrix} -K_i & K_j \end{bmatrix}^\dagger (\hat{\gamma}_i - \hat{\gamma}_j).$$

Substituting κ_i back in $\hat{\gamma}_i$ we have that the vector

$$\hat{\gamma}_i^+ = \hat{\gamma}_i + \begin{bmatrix} K_i & 0 \end{bmatrix} \begin{bmatrix} -K_i & K_j \end{bmatrix}^\dagger (\hat{\gamma}_i - \hat{\gamma}_j)$$

is such that $\mathrm{col}(x_i^{\mathrm{f}}, 0_i^v, 0_i^w) = W_i\hat{\gamma}_i^+$ and $\mathrm{col}(x_j^{\mathrm{f}}, 0_j^v, 0_j^w) = W_j\hat{\gamma}_i^+$. Moreover, we have that $\hat{\gamma}_i^+ \perp (\mathrm{Im}(K_i) \cap \mathrm{Im}(K_j))$, since

$$\begin{bmatrix} \kappa_i \\ \kappa_j \end{bmatrix} \perp \mathrm{Ker}(\begin{bmatrix} -K_i & K_j \end{bmatrix}).$$

We notice that $K_i\kappa_i \perp \mathrm{Im}(K_j)$; by contradiction, if $K_i\kappa_i \not\perp \mathrm{Im}(K_j)$, then one can find $\kappa_i = \tilde{\kappa}_i + \bar{\kappa}_i$, where $K_i\tilde{\kappa}_i \perp \mathrm{Im}(K_j)$ and $K_i\bar{\kappa}_i \in \mathrm{Im}(K_j)$. Let $\bar{\kappa}_j = K_j^\dagger K_i \bar{\kappa}_i$ and $\tilde{\kappa}_j = \kappa_j - \bar{\kappa}_j$. Then $\begin{bmatrix} \bar{\kappa}_i^\top & \bar{\kappa}_j^\top \end{bmatrix}^\top \in \mathrm{Ker}(\begin{bmatrix} -K_i & K_j \end{bmatrix})$ and hence $\begin{bmatrix} \bar{\kappa}_i^\top & \bar{\kappa}_j^\top \end{bmatrix}^\top \not\perp \mathrm{Ker}(\begin{bmatrix} -K_i & K_j \end{bmatrix})$, which contradicts the hypothesis. We conclude that $\begin{bmatrix} K_i & 0 \end{bmatrix} \begin{bmatrix} -K_i & K_j \end{bmatrix}^\dagger (\hat{\gamma}_i - \hat{\gamma}_j) \perp \mathrm{Im}(K_j)$, and, since $\hat{\alpha}_i \perp \mathrm{Im}(K_i)$, we can conclude that $\hat{\gamma}_i^+ \perp (\mathrm{Im}(K_i) \cap \mathrm{Im}(K_j))$. The theorem follows by noticing that after a number of steps equal to the diameter of $\mathcal{G}$, each vector $\hat{\gamma}_i$ verifies all the measurements, since we will have that $\hat{\gamma}_i \perp \bigcap_j^M \mathrm{Im}(K_j)$. Finally, we remark that the solution we are interested in involves only the first N elements of $\hat{\gamma}_i$, corresponding to $\hat{\alpha}_i$, which solves $Y_i\hat{\alpha}_i = x_i^{\mathrm{f}}$. □